ON THE THEORY OF HAMILTONIAN SYSTEMS

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We investigate certain properties of the Hamiltonian systems, connected with the behavior of two functions: of the bilinear form of the canonical variables Ω and of the Poincaré's Function Ω^*. In particular we show that, when a phase point moves along a "straight" path, the elementary Hamiltonian operation represents the total differential of the difference $(\Omega - \Omega^*)$. In the case of periodic orbits we establish that the Hamiltonian operation is multivalued, this being the consequence of the cyclic character of the Poincaré's function Ω^*.

We investigate certain quantities which remain invariant under unrestricted, completely canonical transformations, and indicate the conditions which are necessary for existence of the integrals containing secular terms.

Kepler's problem is used to illustrate the results obtained.

1. General relationships. As we know, the state of a Hamiltonian system is described by the following canonical system of equations:

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \qquad (j=1, 2, \ldots, k) \tag{1.1}$$

Multiplying Eq. (1.1) respectively by p_j and q_j, adding and summing over j, we obtain the differential relation for the bilinear form $\Omega(p, q)$ of the canonical variables p_j and q_j

$$\frac{d\Omega}{dt} = \sum_{j=1}^{k} \frac{\partial H}{\partial p_j} p_j - \sum_{j=1}^{k} \frac{\partial H}{\partial q_j} q_j \qquad \left(\Omega = \sum_{j=1}^{k} p_j q_j\right) \tag{1.2}$$

Following Poincaré [1] we introduce the function Ω^* (Poincaré denotes this function by Ω without the asterisk) by means of the differential relation

$$\frac{d\Omega^*}{dt} = H + \sum_{j=1}^{k} q_j \frac{dp_j}{dt} = H - \sum_{j=1}^{k} \frac{\partial H}{\partial q_j} q_j \tag{1.3}$$

The function Ω^* is defined here with the accuracy of up to an additive constant which depends on the initial state parameters. Eliminating the second term from the right-hand sides of (1.3) and (1.2) and noting that the Hamiltonian $H(t, p, q)$ is defined by

$$H(t, p, q) = \sum_{j=1}^{k} p_j q_j^{\cdot} - L \qquad \left(p_j = \frac{\partial L}{\partial q_j^{\cdot}}\right)$$

we easily obtain

$$d(\Omega - \Omega^*) = \sum_{j=1}^{k} p_j dq_j - Hdt = Ldt \tag{1.4}$$

This enables us to formulate the following theorem.

Theorem 1.1. Let the phase point $N^*(p, q)$ of the dynamic system (1.1) whose Hamiltonian is $H = H(t, p, q)$ move in the $2k$-dimensional phase space E^{2k}. In this case, when the actual motion takes place along a straight path, the elementary Hamiltonian operator Ldt represents the total differential of the difference between the bilinear form Ω of the canonical variables and the Poincaré's function Ω^*.

Corollary 1.1. The condition that

$$\delta(\Omega - \Omega^*)|_{A_1}^{A_2} = 0 \tag{1.5}$$

must hold (necessity), when the representative point of the system moves in the $(k+1)$-dimensional enlarged configuration space $A(q_j, t)$ along a "straight" path from the position $A_1(q_j^{(1)}, t_1)$ to the position $A_2(q_j^{(2)}, t_2)$. This follows directly from the Hamilton's principle by virtue of (1.4).

Theorem 1.2. The converse is also true, i.e. if the condition (1.5) holds along some path, this path is a straight line.

This result is obtained by considering the variation of the difference $(\Omega - \Omega^*)$. Taking into account the fact that the variational process is isochronous ($\delta t = 0$) and, that the variations of the generalized coordinates q_j at the end points are zero ($\delta q_j^{(1)} = 0$, $\delta q_j^{(2)} = 0$), we easily obtain

$$\delta(\Omega - \Omega^*)|_{A_1}^{A_2} = \int_{t_1}^{t_2} \left\{ \sum_{j=1}^{k} \left(q_j^{\bullet} - \frac{\partial H}{\partial p_j} \right) \delta p_j - \sum_{j=1}^{k} \left(p_j^{\bullet} + \frac{\partial H}{\partial q_j} \right) \delta q_j \right\} dt = 0$$

These in turn yield the Hamiltonian equations (1.1), which proves the sufficiency of the condition (1.5).

Relation (1.4) is valid for all reversible and irreversible Hamiltonian systems. It was obtained earlier [1, 2] for the conservative systems, i.e for the cases when the generalized energy integral $H(p, q) = h$ exists.

2. Multivaluedness of the Hamiltonian operator in the case of periodic orbits. Suppose that the Hamiltonian system (1.1) admits a τ-periodic solution $p_j(t)$ and $q_j(t)$ and the phase point $N^*(p, q)$ executes a periodic motion along a closed curve (c). Let us find the Hamiltonian operator for the motion of the phase point $N^*(p, q)$ along the cycle (c). Integrating (1.4) and noting that

$$p_j(t+\tau) = p_j(t), \quad q_j(t+\tau) = q_j(t), \quad \Omega(t+\tau) = \Omega(t)$$

we obtain

$$\oint_{(c)} L dt = -\alpha \qquad (\alpha = \Omega^*(t+\tau) - \Omega^*(t)) \tag{2.1}$$

which can be also written as

$$\int_{0}^{\tau} \sum_{j=1}^{k} p_j dq_j - H dt = -\alpha \tag{2.2}$$

Thus the Hamiltonian operator changes its value by a cyclic constant α during the time τ of a single passage round the circuit (c).

In the natural conservative systems the phase flux moves along the isoenergetic surface

$$H(p, q) = T^*(p, q) + V(q) = h \tag{2.3}$$

Here $T^*(p, q)$ is the associated kinetic energy, $V(q)$ is the potential energy and h is the energy constant.

For such systems the circulation Γ over the cycle (c) by virtue of (2.2) and (2.3) is as follows [3]:

$$\Gamma(c) = \oint_{(c)} \sum_{j=1}^{k} p_j dq_j = -\oint_{(c)} q_j dp_j = h\tau - \alpha \tag{2.4}$$

The circulation $\Gamma(c)$ can be expressed in terms of the kinetic energy averaged over the period τ, i.e $\langle T(\tau) \rangle$. Indeed, by (2.1) and (2.3) we have

$$\langle T(\tau)\rangle = \frac{1}{\tau}\int_0^\tau T^*(p(t), q(t))\,dt = \frac{h\tau - \alpha}{2\tau} \tag{2.5}$$

Consequently the circulation $\Gamma(c)$ is equal to

$$\Gamma(c) = \oint_{(c)} \sum_{j=1}^{k} p_j\,dq_j = 2\tau\,\langle T(\tau)\rangle \tag{2.6}$$

In many cases the cyclic constant α, and consequently the circulation $\Gamma(c)$, can be expressed very simply in terms of the energy constant h and the period τ.

Let a conservative system move in a force field whose potential energy $V(q)$ is a homogeneous function of the generalized coordinates q_j of degree n while the associated expression of kinetic energy $T^*(q, p)$ is a quadratic form of the generalized impulses p_j and an $(-\nu)$-th degree homogeneous function of the generalized coordinates q_j. As we know [2], the cyclic constant α is

$$\alpha = \frac{2+\nu-n}{2+\nu+n}\,h\tau \quad (h \neq 0), \qquad \alpha = -2\tau\,\langle T(\tau)\rangle \quad (h=0) \tag{2.7}$$

when the motion is periodic, therefore by (2.4) we have

$$\Gamma(c) = \frac{2n}{2+\nu+n}\,h\tau \quad (h \neq 0), \qquad \Gamma(c) = -\alpha \quad (h=0) \tag{2.8}$$

We shall illustrate this on the Kepler's problem. Let a unit mass $(m = 1)$ attracted by a force situated at O the potential of which is $V = -\mu / r$ (μ is the reduced mass), execute a periodic motion along an elliptic orbit (c). We find the value J assumed by the Hamiltonian operator during a passage along the orbit (c). Using the Lagrangian

$$L = \frac{1}{2}(x^{\cdot 2} + y^{\cdot 2}) + \frac{\mu}{r} = h + \frac{2\mu}{r} \qquad (h<0) \tag{2.9}$$

together with the integral of the surface elements $c\,dt = r^2 d\vartheta$ and the well known relations of the two-body problem [3, 4]

$$p = r(1 + e\cos\vartheta), \qquad h = -\mu/2a, \qquad c = \sqrt{\mu}\,\sqrt{a(1-e^2)}$$

we obtain

$$J = \oint_{(c)} L\,dt = h\tau + \frac{2\mu p}{c}\int_0^{2\pi} \frac{d\vartheta}{1 + e\cos\vartheta} \tag{2.10}$$

Here r is the radius vector ϑ is the true anomaly, a is the major semiaxis of the ellipse, e is the eccentricity, c is the area constant and τ is the period.

The integral appearing in the right-hand side of (2.10) is

$$\int_0^{2\pi} \frac{d\vartheta}{1 + e\cos\vartheta} = \frac{2\pi}{\sqrt{1-e^2}}$$

Utilizing the value of the period τ we finally obtain

$$J = h\tau - 4h\tau = -3h\tau \quad (\tau = 2\pi a^{3/2} / \sqrt{\mu})$$

Since J is equal to minus α, then by virtue of (2.3), the circulation $\Gamma(c) = h\tau - \alpha = -2h\tau$.

The same result could be obtained directly from (2.8), noting that in the Kepler's problem we have $\nu = 0$ and $n = -1$.

Theorem 2.1. Let the phase point of a system consisting of N gravitating point masses execute a periodic motion along a certain curve (c), with the energy constant different from zero $(h \neq 0)$.

Then the cyclic constant α and the circulation $\Gamma(c)$ are, respectively,

$$\alpha = \frac{2+m}{2-m} h\tau, \qquad \Gamma(c) = \frac{-2m}{2-m} h\tau \qquad (h \neq 0) \tag{2.11}$$

provided that the force of attraction between the particles is inversely proportional to the $(m+1)$-th power of the distance between them.

This follows directly from (2.7) and (2.8), remembering of course that for the gravitating system in question we must set $\nu = 0$ and $n = -m$.

Corollary 2.1 If $h \neq 0$, no periodic motion is possible for the phase point $N^*(p, q)$ of the gravitating system under consideration at $m = 2$.

When $m = 2$, the phase point $N^*(p, q)$ can only execute a periodic motion of the parabolic type $(h = 0)$ which agrees with the motion of a material point in a central force field [5].

3. Certain invariants under completely canonical transformations. We consider, so called, unrestricted, completely canonical transformation with its functional determinant D different from zero

$$(Q_j, P_j) \rightarrow (q_j, p_j) \qquad \left(D = \frac{\partial(Q_j, P_j)}{\partial(q_j, p_j)} \neq 0\right) \tag{3.1}$$

In addition, the domain G of changing of the variables (p_j, q_j) transforms into the domain G' of changing of the variables (P_j, Q_j) in one-to-one correspondence, and the cycle (c) transforms into the cycle (c').

Since the transformation (3.1) is unrestricted, we can use q_j and Q_j $(j = 1, 2, \ldots, k)$ as the independent variables.

Assuming $V = V(q_j, Q_j)$ as a generating function, we can write the following basic differential relations defining the completely canonical transformation [6]

$$\sum_{j=1}^{k} P_j \, dQ_j = \sum_{j=1}^{k} p_j \, dq_j + dV(q_j, Q_j) \tag{3.2}$$

Here the new Hamiltonian function $H'(t, P, Q)$ is obtained from the original function $H(t, p, q)$ by simple change of variables (3.1), so that

$$H(t, p, q) = H'(t, P, Q) \tag{3.3}$$

Theorem 3.1. Let the phase point $N^*(p, q)$ of the Hamiltonian system (1.1) moving along an isoenergetic surface $H(p, q) = h$ execute a periodic motion in the phase space E^{2k} over some cycle (c). Then at any completely canonical transformation (3.2) the product of the period τ and the average kinetic energy $\langle T(\tau)\rangle$ is an invariant. This follows directly from the condition of invariance of the circulation $\Gamma(c)$. Hence, it implies by virtue of (2.5) that the quantity $(h\tau - \alpha)$ when $h \neq 0$ and the cyclic constant α when $h = 0$ are also invariants.

4. On certain integrals containing secular terms. Let

$$q_j = q_j(t, \alpha_i), \; p_j = p_j(t, \alpha_i) \qquad (j = 1, 2, \ldots, k) \tag{4.1}$$

be the solutions of some Hamiltonian system (1.1). Here α_i $(i = 1, 2, \ldots, 2k)$ denote the constants of integration. Following Poincaré [1] we introduce $2k$ functions of the form

$$J_i = 2\sum_{j=1}^{k} \frac{\partial p_j}{\partial \alpha_i} q_j + \sum_{j=1}^{k} p_j \frac{\partial q_j}{\partial \alpha_i} = \frac{\partial \Omega}{\partial \alpha_i} + \sum_{j=1}^{k} q_j \frac{\partial p_j}{\partial \alpha_i} \tag{4.2}$$

Differentiating with respect to t, by virtue of (1.1) and (1.2), we obtain

$$\frac{dJ}{dt}=\frac{\partial}{\partial\alpha_i}\left(\frac{d\Omega}{dt}\right)+\sum_{j=1}^{k}\frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial\alpha_i}-\sum_{j=1}^{k}q_j\frac{\partial}{\partial\alpha_i}\left(\frac{\partial H}{\partial q_j}\right)$$

Further differentiating (1.3) with respect to α_i

$$\frac{\partial}{\partial\alpha_i}\left(\frac{d\Omega^*}{dt}\right)=\frac{\partial H}{\partial\alpha_i}-\sum_{j=1}^{k}\frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial\alpha_i}-\sum_{j=1}^{k}q_j\frac{\partial}{\partial\alpha_i}\left(\frac{\partial H}{\partial q_j}\right)$$

noting that

$$\frac{\partial H}{\partial\alpha_i}=\sum_{j=1}^{k}\frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial\alpha_i}+\sum_{j=1}^{k}\frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial\alpha_i}\tag{4.3}$$

and simplifying, we obtain

$$\frac{dJ_i}{dt}=\frac{\partial}{\partial\alpha_i}\left(\frac{d}{dt}(\Omega+\Omega^*)\right)\tag{4.4}$$

Let us introduce additional $2k$ functions of the form

$$J_i^*=\sum_{j=1}^{k}p_j\frac{\partial q_j}{\partial\alpha_i}\qquad(i=1,2,\ldots,2k)\tag{4.5}$$

Differentiating J_i^* with respect to t by virtue of (1.1), we obtain

$$\frac{dJ_i^*}{dt}=\sum_{j=1}^{k}p_j\frac{\partial}{\partial\alpha_i}\left(\frac{\partial H}{\partial p_j}\right)-\sum_{j=1}^{k}\frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial\alpha_i}$$

If we now differentiate (1.2) with respect to α_i and use (4.3), after simplification we arrive at

$$\frac{dJ_i^*}{dt}=\frac{\partial}{\partial\alpha_i}\left(\frac{d}{dt}(\Omega-\Omega^*)\right)\tag{4.6}$$

which by virtue of (1.4) can be written as

$$\frac{dJ_i^*}{dt}=\frac{\partial L}{\partial\alpha_i}\qquad(i=1,2,\ldots,2k)\tag{4.7}$$

Formulas (4.4) and (4.6) on integration readily yield

$$J_i+J_i^*=2\frac{\partial\Omega}{\partial\alpha_i}+\text{const},\qquad J_i-J_i^*=2\frac{\partial\Omega^*}{\partial\alpha_i}+\text{const}\tag{4.8}$$

where the constants appearing in the right-hand sides depend on the values of the constants $(\alpha_1,\alpha_2,\ldots,\alpha_{2k})$.

Theorem 4.1. Suppose that a conservative system moves in a force field whose potential $V(q)$ is an nth degree homogeneous function of the generalized coordinates q_j while the associated kinetic energy $T^*(q,p)$ is a quadratic form of the generalized impulses p_j, and a homogeneous function of $(-\nu)$-th degree in the generalized coordinates q_j. Then, if the condition $1+n+\nu=0$ holds, integrals of the form

$$J_i=(1-2n)\beta_i t+\text{const}\qquad(\beta_i=\partial H/\partial\alpha_i)\tag{4.9}$$

containing secular terms exist provided that the energy constant h is not zero.

Indeed, we know that if the conditions of homogeneity of the functions $V(q)$ and $T^*(q,p)$ cited above hold, then we have the following relation [2]

$$d/dt(\Omega+\Omega^*)=(1-2n)h+2(1+n+\nu)T^*\qquad(H(p,q)=h)\tag{4.10}$$

Further, utilizing the condition $1+n+\nu=0$ and noting that $H(p(t,\alpha_i),$

$q(t, \alpha_i)) = H^*(\alpha_1, \ldots, \alpha_{2k}) = h$ we obtain, on integrating (4.10), integrals of the form (4.9) which can be written as

$$\frac{\partial \Omega}{\partial \alpha_i} + \sum_{j=1}^{k} q_j \frac{\partial p_j}{\partial \alpha_i} = (1 - 2n)\beta_i t + \text{const} \qquad (h \neq 0) \tag{4.11}$$

When the motion is parabolic ($h = 0$), the secular terms vanish and the integrals become

$$\frac{\partial \Omega}{\partial \alpha_i} + \sum_{j=1}^{k} q_j \frac{\partial p_j}{\partial \alpha_i} = \text{const} \qquad (h = 0) \tag{4.12}$$

We note that the condition $1 + n + \nu = 0$ holds for an unrestricted system consisting of N point masses gravitating according to the Newton's Law, since in this case we have $\nu = 0$ and $n = -1$. Consequently by Theorem 4.1 it follows that when $h \neq 0$, integrals of the type

$$J_i = 3\beta_i t + \text{const} \qquad \left(\beta_i = \frac{\partial H}{\partial \alpha_i},\ i = 1, 2, \ldots, 2k\right) \tag{4.13}$$

exist, which agrees with the result obtained earlier by Poincaré [1].

We can therefore write the following Hamiltonian for the Kepler's problem discussed previously, using the Cartesian coordinates as generalized coordinates $q_1 = x$ and $q_2 = y$. We then have $H = {}^1/_2 (p_x^2 + p_y^2) - \mu / r = h \qquad (r = \sqrt{x^2 + y^2})$

Using further (1.2), (1.3) and the Euler's theorem on homogeneous functions, we easily obtain

$$\frac{d\Omega}{dt} = 2h + \frac{\mu}{r}, \qquad \frac{d\Omega^*}{dt} = h - \frac{\mu}{r}$$

which in turn leads to

$$d/dt\,(\Omega + \Omega^*) = 3h \quad (h = -\mu/2a)$$

The latter expression and (4.4) on integrating and differentiating with respect to parameter α_i yield integrals of the type (4.13).

In particular, for the Kepler's problem discussed above we can set $\alpha_1 = a$, $\alpha_2 = e$ and take into account the fact that

$$x = r\cos\vartheta = a(\cos E - e), \quad y = r\sin\vartheta = a\sqrt{1 - e^2}\sin E$$

(E is the eccentric anomaly), thus obtaining without difficulty

$$p_x = -\mu^{1/2} a^{-1/2} \frac{\sin E}{1 - e\cos E}, \quad p_y = \mu^{1/2} a^{-1/2} (1 - e^2)^{1/2} \frac{\cos E}{1 - e\cos E}, \quad \Omega = e a^{1/2} \mu^{1/2} \sin E$$

The integrals (4.13) in this case become

$$J_1 = x\frac{\partial p_x}{\partial a} + y\frac{\partial p_y}{\partial a} + \frac{\partial \Omega}{\partial a} = \frac{3}{2}\mu a^{-2} t \quad \left(\beta_1 = \frac{\partial h}{\partial a} = \frac{\mu}{2a^2}\right)$$

$$J_2 = x\frac{\partial p_x}{\partial e} + y\frac{\partial p_y}{\partial e} + \frac{\partial \Omega}{\partial e} = 0 \quad \left(\beta_2 = \frac{\partial h}{\partial e} = 0\right)$$

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